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ON A 'GOOD' DENSE CLASS OF TOPOLOGICAL SPACES

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Introduction

Isbell introduced in [8] the concept of topological topology in order to classify adjoint endofunctors of Top (= category of topological spaces and continuous maps). The first author used Isbell's result in [13] and [14], to characterize monoidal closed and biclosed structures on Top in terms of 'good functorial choices' of topological topologies (for any topological space). Since any cocontinuous functor is completely determined by its value on a dense subcategory, then, there is a strict connection between the problem of determining the number of monoidal biclosed structures of Top, and that of describing all topological topologies for spaces of a dense subcategory of Top.

Our aim is to show the existence of a 'good' dense subcategory $\bar{L}$ of Top, with the property that any space in $\bar{L}$ admits the pointwise topology as the finest topological topology.

As a simple consequence, we obtain a result of unicity for monoidal biclosed structures on Top.

Finally, we prove that for a large class of $\bar{L}$-spaces, the Scott topology is strictly finer than the compact-open (= pointwise) topology. Other Hausdorff examples of this fact do not seem to be known.

The paper is organized as follows. In Section 1 we recall the concept of topological (= Isbell) topology and state some basic properties; in Section 2 we establish the connection between topological topologies and biclosed structures; in Section 3 we prove the main result, showing the existence of a dense class of topological spaces for which the Isbell topologies are suitably classificable, and, finally, in Section 4 the comparison of the finest Isbell topology to the Scott topology, for any $\mathbb{N}$-incomplete uniform ultraspace of $\bar{L}$, is made.

After this research was completed, we knew that in a forthcoming paper [3], Činčura obtained, with different techniques, related results to those of Section 2, in the case of any epireflective subcategory of Top.

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1. Isbell topologies

For any topological space X, we shall denote by τ the lattice of open sets of X, by $A, B, C, \ldots$ elements of τ and by τ^* an arbitrary topology on τ. Further, for any set T, we shall write $\langle T \rangle$ to denote the filter of neighbourhoods of T, $\{B \in \tau : B \supseteq T\}$.

1.1. Definition [8].τ^* is an *Isbell topology* (or *topological topology*) iff it makes arbitrary union and finite intersection, continuous maps.

Examples. (a) Let $X \in \mathrm{Top}$ and $\mathscr{F} = \{F\}$ be a family of filters of open sets. If any $F \in \mathscr{F}$ is a compact filter (i.e. $\bigcup_{i\in I} A_i \in F \Rightarrow \bigcup_{i\in J} A_i \in F$, $J \subseteq I$ and J a finite set), and the following condition is satisfied:

$$A \cup B \in F \Rightarrow \text{there exist filters } F_1, F_2, \text{ finite intersection of elements of } \mathscr{F}, \text{ such that } A \in F_1,\ B \in F_2 \text{ and } F_1 \cap F_2 \subseteq F,$$

then the topology $\tau^*_{\mathscr{F}}$, having $\mathscr{F}$ as an open subbase, is a topological topology on τ [13], [17].

(b) If $\mathscr{F} = \{\langle K \rangle\}_{K \in \mathscr{K}}$, where $\mathscr{K}$ is a family of compact subsets of X, verifying the following condition:

$$A \cup B \supseteq K,\ K \in \mathscr{K} \Rightarrow \text{there exist } K_1, K_2, \text{ finite unions of elements of } \mathscr{K}, \text{ such that } K_1 \subseteq A,\ K_2 \subseteq B \text{ and } K \subseteq K_1 \cup K_2,$$

then $\tau^*_{\mathscr{F}}$ is an Isbell topology.

Of course (b) is a particular case of (a); if X is an Hausdorff space, (a) and (b) are equivalent [8].

Families $\mathscr{K}$, satisfying conditions of (b), are given in the following examples (c), (d) and (e).

(c) $\mathscr{K} = \{$all compact subsets of (X, τ_1), with τ_1 a T_2-topology finer than $\tau\}$. If X is an Hausdorff space and $\tau_1 = \tau$, then $\mathscr{K} = \{$all compact subsets of $X\}$ and the associated Isbell topology is the compact-open topology $\tau^*_{\mathrm{c.o.}}$.

(d) A particular case of (c) is $\mathscr{K} = \{$all finite subsets of $X\}$. The associated Isbell topology is the pointwise topology τ^*_{p}.

(e) $\mathscr{K} = \{\{x_0\}\}$, for some $x_0 \in X$. If X is the singleton space $\{*\}$, then the Isbell topology determined by $\mathscr{K} = \{\{*\}\}$, is the Sierpinski topology.

(f) If $\mathscr{F}$ is the empty family, then the associated Isbell topology is the indiscrete one.

1.2. Proposition. *A topology τ^* on τ is an Isbell topology iff it verifies the following conditions.*

I_1: *The union map* $\bigcup$, *and the intersection map* $\bigcap$, *are continuous from* $(\tau, \tau^*) \times (\tau, \tau^*)$ *to* (τ, τ^*).

$I_{2(a)}$: *For any* $\mathscr{A} \in \tau^*$, *if* $\bigcup_{i \in I} A_i \in \mathscr{A}$, *then there exists a finite subset* $J \subseteq I$ *such that* $\bigcup_{i \in J} A_i \in \mathscr{A}$.

$I_{2(b)}$: *For any* $\mathscr{A} \in \tau^*$, $A \in \mathscr{A}$, *if* $B \supseteq A$ *then* $B \in \mathscr{A}$.

Proof. Let τ^* be an Isbell topology; in order to prove $I_{2(a)}$, consider $\mathscr{A} \in \tau^*$ and $\bigcup_{i \in I} A_i \in \mathscr{A}$. From continuity of the arbitrary union map $\bigcup$, it follows that there exists a base neighbourhood $\mathscr{C} = \prod_{i \in I} \mathscr{C}_i$ of $(A_i)_{i \in I}$ in the product topology $\prod_{i \in I} \tau_i^*$ ($\tau_i^* = \tau^*$ for any i), such that, if $B_i \in \mathscr{C}_i$, then $\bigcup_{i \in I} B_i \in \mathscr{A}$; further $\mathscr{C}_i \in \tau^*$, $A_i \in \mathscr{C}_i$ for any $i \in I$ and $\mathscr{C}_i \neq \tau$ only for a finite number of indexes (we denote by J this finite subset of I). Since the family $(B_i)_{i \in I}$, with $B_i = A_i$ for $i \in J$ and $B_i = \emptyset$ for $i \notin J$, is in $\mathscr{C}$, then it follows that $\bigcup_{i \in I} B_i = \bigcup_{i \in J} A_i \in \mathscr{A}$, so $I_{2(a)}$ follows.

Now, in order to prove $I_{2(b)}$ let A be in $\mathscr{A}$ and $B \supseteq A$. Of course all the previous analysis holds for $(A_i)_{i \in I}$, with $A_i = A$, for any $i \in I$, and I a fixed infinite set. We always denote by $\mathscr{C}$ the associated neighbourhood; then $(B_i)_{i \in I}$, with $B_i = A$ if $i \in J$ and $B_i = B$ if $i \notin J$, is in $\mathscr{C}$, so $B = \bigcup_{i \in I} B_i$ is in $\mathscr{A}$.

Conversely, let τ^* be a topology on τ, such that I_1, $I_{2(a)}$ and $I_{2(b)}$ are satisfied; by induction we get the continuity of finite union and intersection, so what we still have to prove is that, given any infinite set I, the union map

$$\bigcup : \prod_{i \in I} (\tau_i, \tau_i^*) \to (\tau, \tau^*)$$

with $(\tau_i, \tau_i^*) = (\tau, \tau^*)$ for any i, is continuous.

If $\mathscr{A} \in \tau^*$ and $\bigcup_{i \in I} A_i \in \mathscr{A}$, then by $I_{2(a)}$ there exists a finite subset J of I such that $\bigcup_{i \in J} A_i \in \mathscr{A}$. Consider the union map

$$\bar{\bigcup} : \prod_{i \in J} (\tau_i, \tau_i^*) \to (\tau, \tau^*)$$

with $(\tau_i, \tau_i^*) = (\tau, \tau^*)$ for any i.

Since we know that $\bar{\bigcup}$ is continuous, there exists a base neighbourhood $\mathscr{C}' = \prod_{i \in J} \mathscr{C}_i'$ of $(A_i)_{i \in J}$ in the product topology $\prod_{i \in J} \tau_i^*$, $\tau_i^* = \tau^*$, such that $A_i \in \mathscr{C}'$ and, if $B_i \in \mathscr{C}_i'$ for any $i \in J$, then $\bigcup_{i \in J} B_i \in \mathscr{A}$. Now, we define (for any $i \in I$), $\mathscr{C}_i = \mathscr{C}_i'$ if $i \in J$ and $\mathscr{C}_i = \tau$ if $i \notin J$; then $(A_i)_{i \in I} \in \mathscr{C} = \prod_{i \in I} \mathscr{C}_i$; further if $(B_i)_{i \in I} \in \mathscr{C}$, then $\bigcup_{i \in J} B_i \in \mathscr{A}$ because $(B_i)_{i \in J} \in \mathscr{C}'$ and therefore by $I_{2(b)}$, $\bigcup_{i \in I} B_i \in \mathscr{A}$. $\square$

1.3. Definition. A subset $\mathscr{A} \subseteq \tau$ is said to verify the condition (C) iff, for any family $(A_i)_{i \in I}$, $I \neq \emptyset$, and any $A \in \mathscr{A}$, if $\bigcup_{i \in I} A_i \supseteq A$, then there exists a finite nonempty subset $J \subseteq I$, such that $\bigcup_{i \in J} A_i \in \mathscr{A}$.

We shall say that τ^* verifies the compactness condition iff any $\mathscr{A} \in \tau^*$ satisfies (C).

Since the conditions $I_{2(a)}$ and $I_{2(b)}$ are clearly equivalent to the compactness condition for τ^*, then, from Proposition 1.2., it follows:

1.4. Corollary. *A topology τ^* on τ is an Isbell topology iff the conditions* I_1 *and*
I_2: *τ^* verifies the compactness condition*
are satisfied.

We note that $\mathrm{I}_{2(a)}$ and $\mathrm{I}_{2(b)}$ are preserved under finite intersection and arbitrary union. Therefore it is enough to check them on a subbase of τ^*. This also means that the set of all $\mathscr{A} \subseteq \tau$ which verify I_2 is a topology on τ; it is the Scott topology of [5] and [16] (denoted by Ω). Therefore, the Corollary 1.4 can be restated saying that an Isbell topology is precisely a topology on τ which verifies I_1 and is coarser than Ω. So, as an obvious consequence, Ω is finer than any Isbell topology; we know that Ω is not in general an Isbell topology (i.e., it does not satisfy I_1) [8].) We shall also provide an example in Section 4.

2. Biclosed structures on the category of topological spaces

We recall that a monoidal category $(\mathsf{V}, \otimes, I, a, l, r)$ [6], is said to be biclosed if every $-\otimes X$, and every $X\otimes -$, has a right adjoint, respectively denoted by $[X,-]$ and by $[\![X,-]\!]$.

When V is symmetric, it is biclosed if closed, with $[X,-]=[\![X,-]\!]$ (see also [11]).

If $\mathsf{V}=\mathsf{Top}$, applying Isbell's Theorem on adjoint endofunctors of Top [8], we get that any monoidal biclosed structure determines a pair of functors $[-,2], [\![-,2]\!] : \mathsf{Top}^0 \to \mathsf{Top}$ (2 is a Sierpinski space), such that, for any $X \in \mathsf{Top}$, $[X,2]$ and $[\![X,2]\!]$ are topological spaces on the set $\tau = \{\text{open sets of } X\}$, and their topologies are Isbell topologies. (For more details see also [13].)

It is known by [1] and [7] that, in the case of Top, a proper class of different monoidal closed structures can be produced. What about biclosed structures? To give an answer to this question, we first state the following.

2.1. Definition. A (full) subcategory L of Top is *dense* in Top iff the coreflective hull of L in Top is Top itself ([11] and [12]).

It is known [1, Theorem 5.2, p. 45], that the coreflective hull of any nonempty $\mathsf{L} \subseteq \mathsf{Top}$ can be characterized as the class of all quotients of topological sums of spaces in L.

Since the internal hom $[-,2]$ of a biclosed structure, carries colimits (and their colimiting cones) to limits (and limiting cones), then if L is dense in Top, for any $X \in \mathsf{Top}$, we obtain

$$[X,2] \cong [\varinjlim_i Y_i, 2] \cong \varprojlim_i [Y_i, 2]$$

with $Y_i \in \mathsf{L}$ for any i; so the functor $[-,2]: \mathsf{Top}^0 \to \mathsf{Top}$ is completely determined by its value on L. Since, by [8, Theorem 1.4, p. 324], the Isbell topologies $[X,2]$ (for any $X \in \mathsf{Top}$) are sufficient to reconstruct the biclosed structure, then the following holds.

2.2. Proposition. *Any monoidal biclosed structure on* Top *is completely determined by the restriction of the functor* $[-,2]$ *to a dense subcategory of* Top.

Consider now a dense class $\bar{\mathsf{L}}$, with the further property that, for any $Y \in \bar{\mathsf{L}}$, there is no Isbell topology on $\tau = \{\text{open sets of } Y\}$ finer than the pointwise. In Section 3 we will show that such a class $\bar{\mathsf{L}}$ exists. The 'good' properties of $\bar{\mathsf{L}}$ imply the following.

2.3. Theorem. *There is exactly one monoidal biclosed structure on* Top.

Proof. Let $(\mathsf{Top}, \otimes, [-,-], [\![-,-]\!])$ be a biclosed structure on Top; we denote by $X \square Y$ the product set of X and Y with the separate continuity topology, and by $[X,Z]_p$ the set $\mathrm{Hom}_{\mathsf{Top}}(X,Z)$ with the pointwise topology. It is easy to see [15, p. 2], that for any $X, Y \in \mathsf{Top}$, $X \square Y$ has a finer topology than $X \otimes Y$. Then, because of the adjunctions $- \otimes X \dashv [X,-]$, $X \otimes - \dashv [\![X,-]\!]$ and $- \square X \cong X \square - \dashv [X,-]_p$, it follows that $[X,Z]$ and $[\![X,Z]\!]$ have topologies which are finer than the pointwise, for any $X, Z \in \mathsf{Top}$. If $Z = 2$ and $X \in \bar{\mathsf{L}}$, we get $[X,2] = [\![X,2]\!] = [X,2]_p$ where $[X,2]_p$ is the pointwise Isbell topology on the set of open sets of X.
By Proposition 2.2, the result follows. $\square$

2.4. Corollary. *There is exactly one symmetric monoidal closed structure on* Top.

Of course, the unique structure of 2.3 and 2.4 is the canonical structure $(\mathsf{Top}, \square, [-,-]_p)$.

3. The dense class of uniform ultraspaces

For any infinite cardinal α and filter $\mathscr{F}$ (or ultrafilter $\mathscr{U}$) on α, the associated filter-space $(X, \mathscr{F})$ (or ultraspace $(X, \mathscr{U})$) is defined as follows: $X = \alpha \cup \{*\}$; $\{x\}$ is open for all $x \in \alpha$, and $\{A \cup \{*\} : A \in \mathscr{F} \text{ (or } \mathscr{U})\}$ is the filter of all neighbourhoods of $*$.

If $\mathscr{F}$ is uniform, then $(X, \mathscr{F})$ is called a uniform filter-space (or uniform ultraspace).

3.1. Definition. $\bar{\mathsf{L}} = \{$all uniform ultraspaces $(X, \mathscr{U})$ on infinite cardinals α, with $\alpha > \omega\}$.

3.2. Proposition. $\bar{\mathsf{L}}$ *is a dense subcategory of* Top.

Proof. It is known, and easy to see, that the coreflective hull of the class $\mathsf{L}_1 = \{$filter-spaces on infinite cardinals $\alpha\}$ is Top. It is also true that any filter-space $(X, \mathscr{F})$ is a quotient of a topological sum of ultraspaces; in fact, if Φ denotes the set of all ultrafilters $\mathscr{U}$, with $\mathscr{U} \supseteq \mathscr{F}$, then the map $f: \sum_{\mathscr{U} \in \Phi} (X, \mathscr{U}) \to (X, \mathscr{F})$, defined by

$$\begin{tikzcd}
\sum_{\mathscr{U} \in \Phi} (X, \mathscr{U}) \arrow[r, tail, "f"] \arrow[d, "i_{\mathscr{U}}"'] & (X, \mathscr{F}) \\
(X, \mathscr{U}) \arrow[ur, "\mathrm{id}"'] &
\end{tikzcd}$$

(with $i_{\mathscr{U}}$ the canonical injections), is a quotient map (if $A \notin \mathscr{F}$, then there exists an ultrafilter $\mathscr{U} \supseteq \mathscr{F}$ with $A \notin \mathscr{U}$).

What we have still to prove is that any ultraspace $(X, \mathscr{U})$, on an infinite cardinal β, is a quotient of a uniform ultraspace $(Z, \mathscr{H})$ on an infinite cardinal α, with $\alpha > \omega$. Let α be any cardinal $> \beta$ (where $\beta \geq \omega$), and denote by $(Y, \mathscr{C}(\alpha))$ the Fréchet filter-space on $\alpha \cup \{ * \}$.

Then, consider the space $(Z, \mathscr{F}_1) = ((\beta \times \alpha) \cup \{ * \}, \mathscr{U} \times \mathscr{C}(\alpha))$, where $\mathscr{U} \times \mathscr{C}(\alpha)$ is the product defined in [4, p. 156]. If $\mathscr{H}$ is an ultrafilter on $\beta \times \alpha$, with $\mathscr{H} \supseteq \mathscr{F}_1$, and $p_1 : \beta \times \alpha \to \beta$ is the first-projection function, then the map $p_1 : (Z, \mathscr{H}) \to (X, \mathscr{U})$ (with $p_1(*) = *$) is a topological quotient (if $A \neq \mathscr{U}$, then $(\beta \setminus A) \times \alpha \in \mathscr{H}$, so $p^{-1}(A) \notin \mathscr{H}$).

Of course, $\operatorname{card} \beta \times \alpha = \alpha$ and, furthermore, for any $B \subseteq \beta \times \alpha$, if $B \in \mathscr{H}$, then $\operatorname{card} B = \alpha$; in fact, if $\operatorname{card} \beta < \alpha$, then $\operatorname{card} p_2(B) < \alpha$ $(p_2 : \beta \times \alpha \to \alpha)$, hence $\alpha \setminus p_2(B) \in \mathscr{C}(\alpha)$ and $\beta \times (\alpha \setminus p_2(B)) \in \mathscr{H}$, so, since $B \cap (\alpha \setminus p_2(B))$ is empty, $B \in \mathscr{H}$ – a contradiction. Then the result follows. $\square$

In order to prove the main result 3.4, we first state the following.

3.3. Lemma. *Consider $X \in \mathsf{Top}$, $\tau = \{$open sets of $X\}$ and an Isbell topology τ^* on τ. If $\mathscr{A}, \tilde{\mathscr{A}}$ are open (nonempty) sets of τ^* satisfying the property*

$$B, C \in \tilde{\mathscr{A}} \Rightarrow B \cap C \in \mathscr{A},$$

then, for any lower directed family $(A_i)_{i \in I}$ of closed-open sets $A_i \in \tilde{\mathscr{A}}$, and for any $A \supseteq \bigcap_{i \in I} A_i$, it results $A \in \mathscr{A}$.

Proof. Since $X = \bigcap_{i \in I} A_i \cup (\bigcup_{i \in I} X \setminus A_i)$ and $A \supseteq \bigcap_{i \in I} A_i$, then $A \cup (\bigcup_{i \in I} (X \setminus A_i)) = X \in \tilde{\mathscr{A}}$. Since $\tilde{\mathscr{A}}$ verifies $\mathrm{I}_{2(a)}$, there exists a finite subset $J \subseteq I$, such that $A \cup (\bigcup_{i \in J} (X \setminus A_i)) \in \tilde{\mathscr{A}}$. Now, if $j \in I$ is such that $A_j \subseteq A_i$, for any $i \in J$, then $A \cup (X \setminus A_j) \in \tilde{\mathscr{A}}$, so $(A \cup (X \setminus A_j)) \cap A_j = A \cap A_j \in \mathscr{A}$, hence $A \in \mathscr{A}$, for $\mathrm{I}_{2(b)}$. $\square$

3.4. Theorem. *For any $(X, \mathscr{U}) \in \mathsf{L}$, the pointwise Isbell topology τ_p^* on $\tau = \{$open sets of $X\}$, is the finest Isbell topology.*

Proof. Let τ^* be an arbitrary Isbell topology on τ; we shall prove that $\tau^* \subseteq \tau_p^*$.

Consider $\mathscr{A} \in \tau^*$ and $A \in \mathscr{A}$. What we have to show is the existence of a finite subset F of A such that $\langle F \rangle \subseteq \mathscr{A}$.

If $\{*\}\notin A$, since τ^* verifies the compactness condition and $A=\bigcup_{x\in A}\{x\}$, with $\{x\}\in\tau$, then $\bigcup_{x\in F}\{x\}\in\mathscr{A}$, where F is a finite subset of A, so $A\in\langle F\rangle\subseteq\mathscr{A}$.

Let us consider now the case in which $\{*\}\in A$. We choose a lower directed sequence $\{\mathscr{A}_n\}_{n\in\mathbb{N}}$, $\mathscr{A}_n\in\tau^*$, such that:

- $\mathscr{A}_0=\mathscr{A}$,
- for any n, $\mathscr{A}_{n+1}$ is an open neighbourhood of A which verifies the condition: $B,C\in\mathscr{A}_{n+1}\Rightarrow B\cap C\in\mathscr{A}_n$.

The possibility of such a choice is trivial consequence of continuity of intersection.

We suppose that, for any $B\in\mathscr{A}_1$, $\{*\}\in B$; in fact, if there exists $B\in\mathscr{A}_1$ with $\{*\}\notin B$, then there also exists a finite set $F\subseteq B$, with $F\in\mathscr{A}_1$, so $A\cap F\in\mathscr{A}$, hence $A\in\langle A\cap F\rangle\subseteq\mathscr{A}$.

Denote by $\mathscr{A}'$ the set $\bigcap_{n\in\mathbb{N}}\mathscr{A}_n$, it is easy to see that $\mathscr{A}'$ is a filter; furthermore, if we consider the set $F'=\bigcap_{C\in\mathscr{A}'}C$, then $\mathscr{A}'=\langle F'\rangle$. In fact $\mathscr{A}'\subseteq\langle F'\rangle$ by definition of F' and, conversely, if $B\supseteq F'$, then, since any $C\in\mathscr{A}'$ is closed and open ($\{*\}\in C$), by Lemma 3.3 it follows that $B\in\mathscr{A}_n$ for any n, so $B\in\mathscr{A}'$.

We prove now that the set F' is a Lindelöf subspace of X. If $\bigcup_{i\in I}A_i\supseteq F'$, then $\bigcup_{i\in I}A_i\in\mathscr{A}_n$, $\forall n$, so there exists, for any n, a finite set $I_n\subseteq I$ such that $\bigcup_{i\in I_n}A_i\in\mathscr{A}_n$; putting $I'=\bigcup_{n\in\mathbb{N}}I_n$, where I' is obviously a countable set, it follows that $\bigcup_{i\in I'}A_i\in\mathscr{A}_n$ for any n, hence $\bigcup_{i\in I'}A_i\in\mathscr{A}'$ and so $\bigcup_{i\in I'}A_i\supseteq F'$. It is easy to see that any Lindelöf subspace of X is necessarily countable (it suffices to divide F' into two parts with cardinality $>\omega$ and to use properties of ultrafilters).

Now, define $\bar{A}=X\setminus(F'\setminus\{*\})$; since $\alpha>\omega$, then $\bar{A}\in\mathscr{C}(\alpha)$ (= Fréchet filter on α) and, $\mathscr{U}$ being an uniform ultrafilter, $\bar{A}\in\mathscr{U}$.

Since $X=\bar{A}\cup(\bigcup_{x\in F'\setminus\{*\}}\{x\})\in\mathscr{A}_1$, there exists a finite set $\tilde{F}\subseteq F'\setminus\{*\}$, such that $\bar{A}\cup(\bigcup_{x\in\tilde{F}}\{x\})\in\mathscr{A}_1$. If $F=\tilde{F}\cup\{*\}$, then $\langle F\rangle\subseteq\mathscr{A}$. To prove this fact, suppose that $B\supseteq F$. Then the open set $B\cup(F'\setminus\{*\})$, containing F', is in $\mathscr{A}'$, hence is also in $\mathscr{A}_1$; since $\bar{A}\cup\tilde{F}\in\mathscr{A}_1$, the intersection open set

$$(\bar{A}\cup\tilde{F})\cap(B\cup(F'\setminus\{*\}))=(\bar{A}\cap B)\cup(\tilde{F}\cap B)\cup\tilde{F}$$

is in $\mathscr{A}_0=\mathscr{A}$. For $B\supseteq(\bar{A}\cap B)\cup(\tilde{F}\cap B)\cup\tilde{F}$, then $B\in\mathscr{A}$; furthermore, since $A\in\mathscr{A}'$, $A\supseteq F'\supseteq F$ and so $A\in\langle F\rangle\subseteq\mathscr{A}$. □

3.5. Remark. Note that we have proved 3.4 without using the continuity of the union map.

4. A remark on Scott topologies for some uniform ultraspaces

In Section 3, we proved that, for any $(X,\tau)\in\bar{\mathrm{L}}$, and any Isbell topology τ^* on τ, we have $\tau^*\subseteq\tau^*_{\mathrm{p}}$. Remark 3.5 leaves open the question if the continuity assumption for the intersection map is necessary for the thesis. Isbell in [8] and [10] proposed a similar problem in terms of Scott topologies.

It is known that, for any $(X,\tau)\in\mathrm{Top}$, there is a finest Isbell topology on τ (always contained in the Scott topology) and that in 'nice cases' it is just the compact-open topology (see [8] and [10]); we have proved in Section 3 that this is the case for any $X\in\bar{\mathrm{L}}$.

It is clear that if Theorem 3.4 holds without the continuity of the intersection, then it would actually say that the Scott topology and the pointwise coincide. Therefore the fact that the continuity assumption of the finite intersection map cannot be dropped means precisely that the Scott topology is strictly finer than the pointwise topology.

We prove that this last statement is true for a large class of ultraspaces, namely for all $\mathbb{N}$-incomplete spaces of $\bar{\mathrm{L}}$, producing in this way examples of spaces for which the Scott topology does not verify condition I_1 of 1.2. We do not know any other Hausdorff example of this fact.

Let us recall that an ultraspace $(X,\mathscr{U})$ is said to be $\mathbb{N}$-incomplete if $\mathscr{U}$ is $\mathbb{N}$-incomplete [4], i.e., if there exists a family $\{E_n\}_{n\in\mathbb{N}}\subseteq\mathscr{U}$ such that $\bigcap_{n\in\mathbb{N}}E_n=\emptyset$.

4.1. Proposition. *For any $\mathbb{N}$-incomplete uniform ultraspace $(X,\mathscr{U})$ on a cardinal $\alpha>\omega$, the Scott topology Ω is strictly finer than the pointwise topology τ_{p}^*.*

Proof. Denote by $\{E_n\}_{n\in\mathbb{N}}$ a sequence of elements of $\mathscr{U}$, such that $\bigcap_{n\in\mathbb{N}}E_n=\emptyset$ and $\mathrm{card}(X\setminus E_1)\geq\omega$. Consider the subset $\mathscr{A}$ of τ defined by:

$$A\in\mathscr{A}\quad\Leftrightarrow\quad\begin{array}{l}\text{(i) } *\in A,\\ \text{(ii) } \mathrm{card}(A\setminus E_n)\geq n \quad \text{for any } n.\end{array}$$

Since $X\in\mathscr{A}$, $\mathscr{A}$ is not empty. Further, $\mathscr{A}$ trivially verifies the condition $\mathrm{I}_{2(\mathrm{b})}$ of 1.2. In order to prove that $\mathscr{A}$ verifies $\mathrm{I}_{2(\mathrm{a})}$, consider $\bigcup_{i\in I}A_i\in\mathscr{A}$, $A_i\in\tau$. Denote by j an I-index such that $*\in A_j$. Since $A_j\setminus\{*\}\in\mathscr{U}$, it is easy to see that the set $\mathscr{N}=\{n\in\mathbb{N},\mathrm{card}(A_j\setminus E_n)\leq\omega\}$ is finite. So, for any $n\notin\mathscr{N}$, we have $\mathrm{card}(A_j\setminus E_n)>n$.

If $\bar{n}\in\mathscr{N}$, $\mathrm{card}(A_j\setminus E_{\bar{n}})\leq\omega$, and since $\bigcup_{i\in I}A_i\in\mathscr{A}$, there exist $\bar{n}$ points $x_1,x_2,\dots,x_{\bar{n}}$ of X such that $x_l\notin E_{\bar{n}}$ and $x_l\in A_{i_l}$, $1\leq l\leq\bar{n}$. Then the set $A_j\cup(\bigcup_{1\leq l\leq\bar{n}}A_{i_l})$ verifies the condition (ii) for any $n\notin\mathscr{N}$ and for $n=\bar{n}$.

Since $\mathscr{N}$ is finite, repeating the same construction for any $\bar{n}\in\mathscr{N}$, we obtain a finite union of A_i verifying (ii) for any $n\in\mathscr{N}$.

So we have proved that $\mathscr{A}\in\Omega$. It suffices now to see that $\mathscr{A}$ is not open in τ_{p}^*. If F is any finite subset of X, then, since for any $n>\mathrm{card}\,F$, the open set $F\cup E_n$ is in $\langle F\rangle$, but not in $\mathscr{A}$, it follows that $\langle F\rangle\not\subset\mathscr{A}$. $\square$

From what we have just proved, the fact that the continuity condition of the intersection map cannot be removed from the hypotheses of 3.4 follows. As a trivial consequence we finally obtain the following

4.2. Corollary. *For any $\mathbb{N}$-incomplete uniform ultraspace on a cardinal $\alpha>\omega$, the Scott topology is not topological.*

We finally point out that the spaces of Corollary 4.2 determine a dense class of Top, since one can repeat the arguments of Section 3 restricting oneself to consider cardinals α of cofinality ω; it is clear that in such a case any uniform ultraspace is necessarily $\mathbb{N}$-incomplete.

Therefore, from Corollary 4.2, we trivially get:

4.3. Corollary. *The class of normal spaces for which the Scott topology is not an Isbell topology is dense in* Top.

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